

# CRITERIA FOR VIRTUAL FIBERING

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ABSTRACT. We prove that an irreducible 3-manifold whose fundamental group satisfies a certain group-theoretic property called *RFRS* is virtually fibered. As a corollary, we show that 3-dimensional reflection orbifolds and arithmetic hyperbolic orbifolds defined by a quadratic form virtually fiber. These include the Seifert Weber dodecahedral space and the Bianchi groups. Moreover, we show that a taut sutured compression body has a finite-sheeted cover with a depth one taut-oriented foliation.

## 1. INTRODUCTION

Thurston proposed the question of whether a hyperbolic 3-manifold has a finite-sheeted cover fibered over  $S^1$  [32]. A connected fibered manifold with fiber of Euler characteristic  $< 0$  must be irreducible. However, there exist non-fibered closed irreducible 3-manifolds which have no cover which fibers over  $S^1$ , the simplest example being a Seifert fibered space with non-zero Euler class and base orbifold of non-zero Euler characteristic [29]. Since Thurston posed this question, many classes of hyperbolic 3-manifolds have been proven to virtually fiber. Thurston proved that the reflection group in a right-angled hyperbolic dodecahedron virtually fibers.

Gabai gave an example of a union of two  $I$ -bundles over non-orientable surfaces, which has a 2-fold cover which fibers, but in some sense this example is too simple since it fibers over a mirrored interval (this example was known to Thurston). He also gave an example of a 2 component link which is not fibered and has a 2-fold cover which fibers, which is not of the previous type of example [13]. Aitchison and Rubinstein generalized Thurston's example to certain cubulated 3-manifolds [3]. Reid gave the first example of a rational homology sphere which virtually fibers using arithmetic methods [30]. Leininger showed that every Dehn filling on one component of the Whitehead link virtually fibers [22]. Walsh showed that all of the non-trivial 2-bridge link complements virtually fiber [36]. There are more recent unpublished results of DeBlois and Agol, Boyer, and Zhang exhibiting classes of Montesinos links which virtually fiber by generalizing Walsh's method. Most of these examples are shown to be virtually fibered by exhibiting a fairly explicit cover of the manifold which fibers. Button found many virtually fibered manifolds in the Snappea census of cusped and closed hyperbolic 3-manifolds [7]. Of course, we have neglected mention of the many papers exhibiting classes of 3-manifolds which fiber.

A characterization of virtual fibering was given by Lackenby [21]. He showed that if  $M$  is a closed hyperbolic manifold, with a tower of regular covers such that the Heegaard genus grows slower than the fourth root of the index of the cover, then  $M$  virtually fibers. It is

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difficult, however, to apply this criterion to a specific example in order to prove it virtually fibers since it is difficult to estimate the Heegaard genus of covering spaces.

In this paper, we give a new criterion for virtual fibering which is a condition on the fundamental group called RFRS. Using this condition, and recent results of Haglund-Wise [15], we prove that hyperbolic reflection groups and arithmetic groups defined by quadratic forms virtually fiber. Rather than state the RFRS condition in the introduction, we state the following theorem, which is a corollary of Theorem 5.1 and Corollary 2.3:

**Theorem 1.1.** *Let  $M$  be a compact irreducible orientable 3-manifold with  $\chi(M) = 0$ . If  $\pi_1(M)$  is a subgroup of a right-angled Coxeter group or right-angled Artin group, then  $M$  virtually fibers.*

The strategy of the proof is intuitive. If an oriented surface  $F \subset M$  is non-separating, then  $M$  fibers over  $S^1$  with fiber  $F$  if and only if  $M \setminus F \cong I \times F$ . If  $F$  is not a fiber of a fibration, then there is a JSJ decomposition of  $M \setminus F$  [18, 19], which has a product part (window) and a non-product part (guts). The idea is to produce a complexity of the guts, and use the RFRS condition to produce a cover of  $M$  to which a component of the guts lifts and for which we can decrease the complexity of the guts, by “killing” it using non-separating surfaces coming from new homology in this cover. The natural setting for these ideas is sutured manifold hierarchies [12], but we choose instead to use least-weight taut normal surfaces introduced by Oertel [28] and further developed by Tollefson and Wang [35]. The replacement for sutured manifold hierarchies is branched surfaces, whose machinery was introduced by Floyd and Oertel [11]. We review some of the results from these papers, but in order to follow the proofs in this paper, one should be familiar with the results of [28, 35].

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## 2. RESIDUALLY FINITE RATIONAL SOLVABLE GROUPS

The rational derived series of a group  $G$  is defined inductively as follows. If  $G^{(1)} = [G, G]$ , then  $G_r^{(1)} = \{x \in G \mid \exists k \neq 0, x^k \in G^{(1)}\}$ . If  $G_r^{(n)}$  has been defined, define  $G_r^{(n+1)} = (G_r^{(n)})_r^{(1)}$ . The rational derived series gets its name because  $G_r^{(1)} = \ker\{G \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} G/G^{(1)}\}$ . The quotients  $G/G_r^{(n)}$  are solvable.

**Definition 2.1.** *A group  $G$  is residually finite  $\mathbb{Q}$ -solvable or RFRS if there is a sequence of subgroups  $G = G_0 > G_1 > G_2 > \dots$  such that  $G \triangleright G_i$ ,  $\cap_i G_i = \{1\}$ ,  $[G : G_i] < \infty$  and  $G_{i+1} \geq (G_i)_r^{(1)}$ .*

By induction,  $G_i \geq G_r^{(i)}$ , and thus  $G/G_i$  is solvable with derived series of length at most  $i$ . We remark that if  $G$  is RFRS, then any subgroup  $H < G$  is as well. Also, we remark that to check that  $G$  is RFRS, we need only find a cofinal sequence of finite index subgroups  $G > G_i$  such that  $G_{i+1} > (G_i)_r^{(1)}$ . This follows because  $G \triangleright \text{Core}(G_i) = \cap_{g \in G} gG_i g^{-1}$ . Then

the sequence  $G \triangleright \text{Core}(G_1) \triangleright \text{Core}(G_2) \triangleright \cdots$  will satisfy the criterion by the following argument. For  $g \in G$ ,  $g((G_i)_r^{(1)})g^{-1} = (gG_i g^{-1})_r^{(1)}$ , so

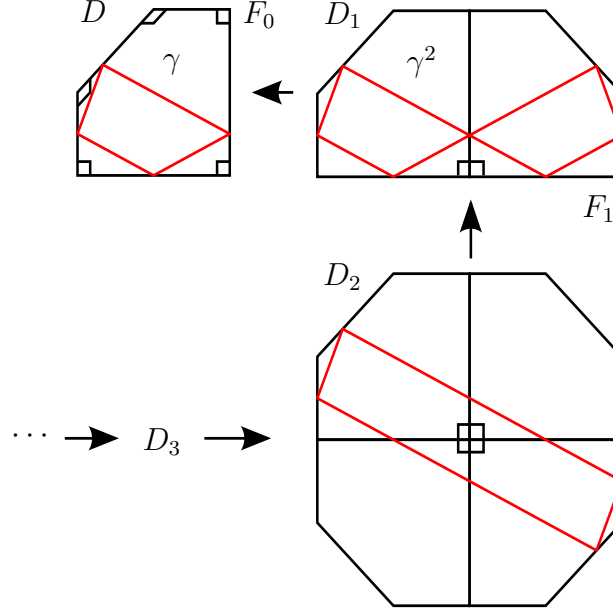
$$\text{Core}(G_{i+1}) = \cap_{g \in G} gG_{i+1}g^{-1} \geq \cap_{g \in G} (gG_i g^{-1})_r^{(1)} \geq (\text{Core}(G_i))_r^{(1)}.$$

Thus, the new sequence satisfies the properties of the RFRS criterion. The relevance of this definition may not be immediately apparent, but it was discovered by the following method. Suppose we have a manifold (or more generally a topological space)  $M$  with  $\pi_1 M = G$  such that  $\text{rk} H_1(M; \mathbb{Q}) > 0$ . Choose a non-trivial finite quotient  $H_1(M; \mathbb{Z})/\text{Torsion} \rightarrow H_0$ , and take the cover  $M_1$  of  $M$  corresponding to  $\ker\{\pi_1(M) \rightarrow H_0\}$ . Suppose that  $\text{rk} H_1(M_1; \mathbb{Q}) > \text{rk} H_1(M; \mathbb{Q})$ . Then we may take a finite index abelian cover  $M_2 \rightarrow M_1$  coming from a quotient of  $H_1(M_1; \mathbb{Z})/\text{Torsion}$  which does not factor through  $H_1(M_1) \rightarrow H_1(M)$ . If we can repeat this process to get a sequence of covers  $M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$  such that  $M_{i+1}$  is a finite abelian cover of  $M_i$ , and the sequence of covers is cofinal (so that the limiting covering space is the universal cover), then  $\pi_1(M)$  is RFRS. Notice that we may assume that  $\text{rk} H_1(M_i; \mathbb{Q})$  is unbounded, unless  $\pi_1(M)$  is virtually abelian. Thus, we see that a manifold  $M$  such that  $\pi_1(M)$  is not virtually abelian but is RFRS has virtual infinite first betti number. This formulation of the RFRS condition is the characterization that we will apply to study 3-manifolds.

**Theorem 2.2.** *A finitely generated right-angled Coxeter group  $G$  has a finite index subgroup  $G'$  such that  $G'$  is RFRS.*

*Proof.* Let  $G$  act on a convex subset  $C \subset \mathbb{R}^n$  with quotient a polyhedral orbifold  $D = C/G$ , such that each involution generator of  $G$  fixes a codimension one hyperplane in  $C$ . This is possible by taking  $C$  to be homeomorphic to the Tits cone for the group  $G$  [5, V.4], [16, 5.13]. We may take a subgroup  $G' < G$  of finite index inducing a manifold cover  $D' \rightarrow D$ , such that for each codimension one face  $F$  of  $D$ , the preimage of  $F$  in  $D'$  is embedded, orientable and 2-sided. For example, we may take  $G' = \ker\{G \rightarrow G/G^{(1)}\}$ , since  $G/G^{(1)} \cong (\mathbb{Z}/2\mathbb{Z})^{\text{rk} G}$ . Consider a cofinal sequence of finite-index covers  $D \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots$  such that  $D_{i+1}$  is a 2-fold cover of  $D_i$  obtained by reflecting in a face  $F_i \subset D_i$  (see Figure 1). Let  $G_i = \pi_1(D_i) \cap G'$ , with quotient manifold  $D'_i = C/G_i$ . We need to show that  $(G_i)_r^{(1)} < G_{i+1}$ . Clearly,  $(G_i)^{(1)} < G_{i+1}$ , since  $G_i/G_{i+1} \cong \mathbb{Z}/2\mathbb{Z}$  or  $G_{i+1} = G_i$ . Thus, we need only show that any element mapping non-trivially to  $G_i/G_{i+1}$  is not torsion, so that we have a factorization  $G_i \rightarrow H_1(G_i)/\text{Torsion} \rightarrow \mathbb{Z}/2\mathbb{Z}$ . Suppose  $g \in G_i - G_{i+1}$ . Take a loop  $\gamma$  representing  $g$  in  $D'_i$ , and project  $\gamma$  down to  $D_i$  generically. We see a closed path bouncing off the faces of  $D_i$  (see Figure 1). Then the projection of  $\gamma$  must hit  $F_i$  an odd number of times, otherwise it would lift to  $D_{i+1}$  and therefore would be contained in  $G_{i+1}$ . We therefore see  $\gamma$  intersecting the preimage of  $F_i$  in  $D'_i$  an odd number of times. Since the preimage of  $F_i$  is an orientable embedded 2-sided codimension 1 submanifold, we see that  $\gamma$  must represent an element of infinite order in  $H_1(G_i) = G_i/(G_i)^{(1)}$ , and therefore  $0 \neq 2[\gamma] \in H_1(D'_i; \mathbb{Q}) = \mathbb{Q} \otimes G_i/(G_i)_r^{(1)}$ . This implies that  $g^2 \notin (G_i)_r^{(1)}$ . But  $g^2 \in G_{i+1}$ , since  $[G_i : G_{i+1}] \leq 2$ . Therefore, we have  $G_i/G_{i+1} \cong \mathbb{Z}/2\mathbb{Z}$  is a quotient of  $H_1(G_i; \mathbb{Z})/\text{Torsion}$ . Thus, we see that  $G'$  is RFRS.  $\square$

**Corollary 2.3.** *The following groups are virtually RFRS:*

FIGURE 1. A right-angled polyhedral orbifold  $D$  and a tower of 2-covers

- *surface groups*
- *reflection groups*
- *right-angled Artin groups*
- *arithmetic hyperbolic groups defined by a quadratic form*
- *direct products and free products of RFRS groups*

*Proof.* Surface groups are subgroups of the reflection group in a right-angled pentagon (see e.g. [31]). Reflection groups, right-angled Artin groups, and arithmetic hyperbolic groups defined by a quadratic form all have finite index subgroups which are subgroups of right-angled reflection groups by [15, p. 3] (this was proven for cusped arithmetic 3-manifolds previously in [1]), and are thus virtually RFRS by Theorem 2.2.

Direct products of RFRS groups are immediately seen to be RFRS. If  $G, H$  are RFRS, then we have  $G_i \triangleleft G$ ,  $[G : G_i] < \infty$ , such that  $(G_i)_r^{(1)} < G_{i+1}$ , and  $H_i \triangleleft H$ ,  $[H : H_i] < \infty$ , such that  $(H_i)_r^{(1)} < H_{i+1}$ . Then  $G_i \times H_i \triangleleft G \times H$  is finite index, and  $(G_i \times H_i)_r^{(1)} = (G_i)_r^{(1)} \times (H_i)_r^{(1)} \triangleleft G_{i+1} \times H_{i+1}$ .

Free products of RFRS groups are also easily seen to be RFRS. Let  $G$  and  $H$  be (non-trivial) RFRS as above, with  $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots$ ,  $H = H_0 \triangleright H_1 \triangleright H_2 \triangleright \dots$ . We construct inductively groups  $GH_i \triangleleft G * H$  such that  $GH_i \cong (H_i)^{*k(i)} * (G_i)^{*j(i)} * \mathbb{Z}^{*l(i)}$ . To obtain  $GH_{i+1}$  from  $GH_i$ , take the kernel of the quotient of  $GH_i$  onto  $(H_i/H_{i+1})^{k(i)} \times (G_i/G_{i+1})^{j(i)} \times (\mathbb{Z}/2\mathbb{Z})^{l(i)}$ . This may be easily seen by induction of the action on the Bass-Serre tree. The free product  $G * H$  is a graph of groups with graph a single edge with trivial stabilizer, and two vertices with stabilizers  $G$  and  $H$ , respectively. The Bass-Serre tree  $T$  is infinite bipartite. When we take the quotient  $T/GH_i$ , we get a graph of groups with bipartite graph, and vertices corresponding to conjugates of  $G_i$  and  $H_i$ , which is how we see the claimed Kurosh decomposition, where the rank of the free factor  $l(i)$  corresponds to

the first betti number of the graph  $T/GH_i$ . For any element  $f \in G * H$ , we may write  $f$  (up to conjugacy) as a cyclically reduced product  $f = g_1 h_1 g_2 h_2 \cdots g_m h_m$ , where  $g_j \neq 1, h_j \neq 1$ , unless  $f$  is conjugate into  $G$  or  $H$ , in which case it is easily seen that  $f \notin GH_i$  for some  $i$ . Choose  $i$  such that  $g_j \notin G_i, h_j \notin H_i, j = 1, \dots, m$ . Then the projection of a loop representing  $f$  to the graph  $T/GH_i$  will be homotopically non-trivial, since each time it enters a vertex of  $T/GH_i$ , it will exit a distinct edge, since  $g_j$  and  $h_j$  are not in the vertex stabilizers of  $T/GH_i$ . Thus, the projection of  $f$  to  $\mathbb{Z}^{*l(i)}$  will be non-trivial, and since the free group is RFRS (with respect to 2-group quotients), we see that  $f$  will eventually not be in  $GH_i$ .  $\square$

### 3. SUTURED MANIFOLDS

We consider 3-manifolds that are triangulated and orientable, and PL curves and surfaces. For a proper embedding  $Y \subset X$ , define  $X \setminus \setminus Y$  to be the manifold  $\overline{X \setminus \mathcal{N}(Y)}$ , where  $\mathcal{N}(Y)$  denotes a regular neighborhood of  $Y$ .

**Definition 3.1.** *Let  $S$  be a compact connected orientable surface. Define  $\chi_-(S) = \max\{-\chi(S), 0\}$ . For a disconnected compact orientable surface  $S$ , let  $\chi_-(S)$  be the sum of  $\chi_-$  evaluated on the connected subsurfaces of  $S$ .*

In [33], Thurston defines a pseudonorm on  $H_2(M, \partial M; \mathbb{R})$  and  $H_2(M; \mathbb{R})$ , and this was generalized by Gabai [12].

**Definition 3.2.** *Let  $M$  be a compact oriented 3-manifold. Let  $K$  be a subsurface of  $\partial M$ . Let  $z \in H_2(M, K)$ . Define the norm of  $z$  to be*

$$x(z) = \min\{\chi_-(S) \mid (S, \partial S) \subset (M, K), \text{ and } [S] = z \in H_2(M, K)\}.$$

Thurston proved that  $x(nz) = nx(z)$ , and therefore  $x$  may be extended to a norm  $x : H_2(M, K; \mathbb{Q}) \rightarrow \mathbb{Q}$ . Then  $x$  is extended to  $x : H_2(M, K; \mathbb{R}) \rightarrow \mathbb{R}$  by continuity.

We have the Poincaré duality isomorphisms  $PD : H^1(M) \rightarrow H_2(M, \partial M)$  and  $PD : H_2(M, \partial M) \rightarrow H^1(M)$ , so that  $PD^2 = Id$ . We may define for  $\alpha \in H^1(M)$ ,  $x(\alpha) = x(PD(\alpha))$ . We say that  $\alpha$  is a fibered cohomology class if  $c \cdot PD(\alpha)$  is represented by a fiber for some constant  $c$ . Denote by  $\mathcal{B}_x(M)$  the unit ball of the Thurston norm on  $H^1(M)$  or on  $H_2(M, \partial M)$ .

**Definition 3.3.** *Let  $S$  be a properly embedded oriented surface in the compact oriented 3-manifold  $M$ . Then  $S$  is taut in  $H_2(M, K)$  if  $\partial S \subset K$ ,  $S$  is incompressible, there is no homologically trivial union of components of  $S$ , and  $\chi_-(S) = x([S])$  for  $[S] \in H_2(M, K)$ .*

**Definition 3.4.** *A taut (codimension 1) foliation is a foliation which has a closed transversal going through every leaf.*

The following theorem says that a compact leaf of a taut oriented foliation is a taut surface. This was proven by Thurston, under the assumption that the foliation is differentiable [33]. Gabai generalized this to deal with general foliations [14].

**Theorem 3.5.** *Let  $M$  be a compact oriented 3-manifold. Let  $F$  be a taut oriented foliation of  $M$  such that  $F$  is transverse to  $\partial M$ . If  $R$  is a compact leaf of  $F$ , then  $R$  is taut.*

If  $M$  is a 3-manifold fibering over  $S^1$  with fiber  $F$ , then there is a cone over a face  $E$  of  $\mathcal{B}_x(M)$  whose interior contains  $[F]$  and such that every other rational homology class in  $\text{int}E$  is also a fibered class [33].

Gabai introduced the notion of a sutured manifold in order to prove the converse of the above theorem.

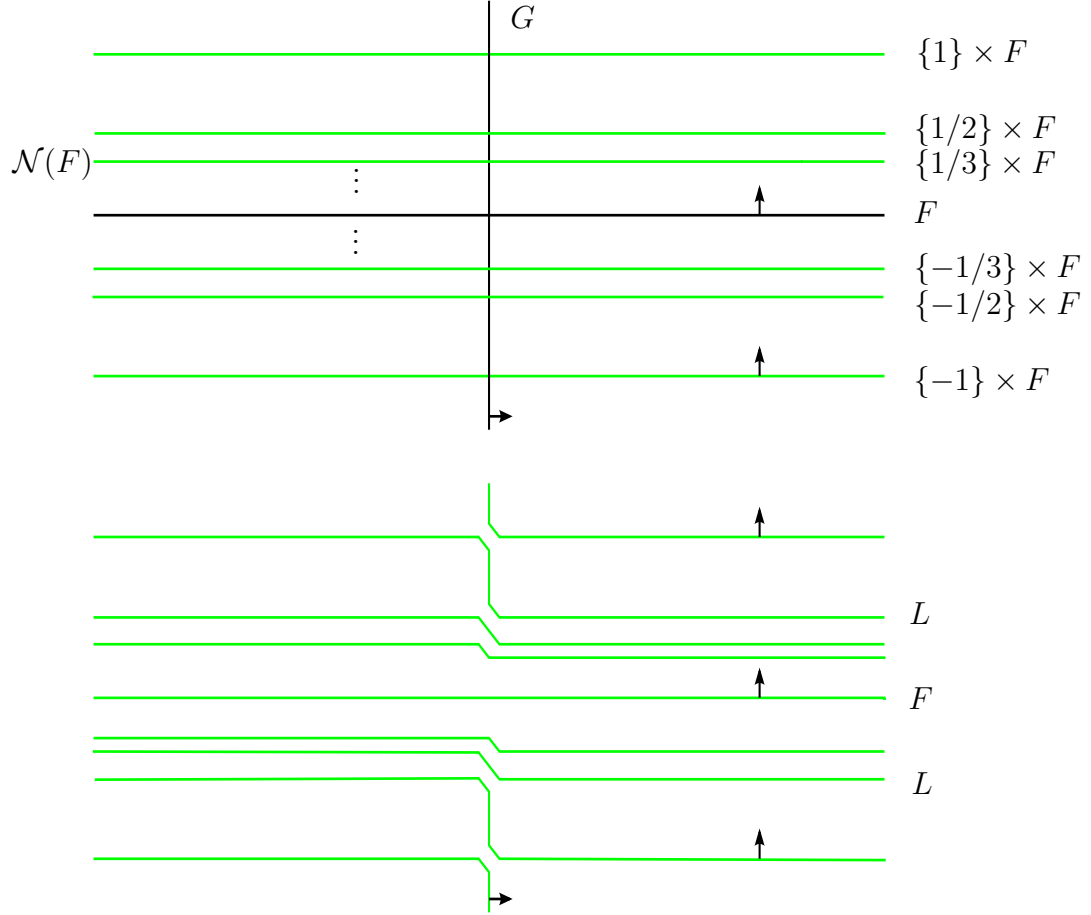
**Definition 3.6.** *A sutured manifold  $(M, \gamma)$  is a compact oriented 3-manifold  $M$  together with a set  $\gamma \subset \partial M$  of pairwise disjoint annuli  $A(\gamma)$  and tori  $T(\gamma)$ . Furthermore, the interior of each component of  $A(\gamma)$  contains a suture, i.e., a homologically nontrivial oriented simple closed curve. We denote the set of sutures by  $s(\gamma)$ . Finally, every component of  $R(\gamma) = \partial M \setminus A(\gamma)$  is oriented. Define  $R_+(\gamma)$  (or  $R_-(\gamma)$ ) to be those components of  $R(\gamma)$  whose normal vectors point out of (into)  $M$ . The orientations on  $R(\gamma)$  must be coherent with respect to  $s(\gamma)$ , i.e., if  $\delta$  is a component of  $\partial R(\gamma)$ , and is given the boundary orientation, then  $\delta$  must represent the same homology class in  $H_1(\gamma)$  as the suture in the component of  $A(\gamma)$  containing  $\delta$ .*

**Theorem 3.7.** *Let  $M$  be a 3-manifold, and let  $f \in H_2(M, \partial M)$  be a homology class which lies in the boundary of the cone over a fibered face  $E$  of  $\mathcal{B}_x(M)$ . Let  $(F, \partial F) \subset (M, \partial M)$  be a taut surface representing  $f$ . Then  $M \setminus \setminus F$  is a taut sutured manifold which admits a depth one taut oriented foliation.*

*Proof.* Let  $G$  be a surface such that  $[G] \in \text{int}E$  is a surface fibering over  $S^1$ . Since  $x([G] + [F]) = x([G]) + x([F])$ , we may make  $G$  and  $F$  transverse so that  $\chi_-(G \oplus F) = \chi_-(G) + \chi_-(F)$ , where  $G \oplus F$  indicates oriented cut and paste. Then  $F \cap (M \setminus \setminus G)$  gives a taut sutured manifold decomposition of  $M \setminus \setminus G$  [12]. Since  $G$  is a fibered surface,  $M \setminus \setminus G$  is a product. Any taut sutured decomposition of a product sutured manifold is also a product sutured manifold, so  $M \setminus \setminus (G \cup F)$  is a product sutured manifold, with sutures corresponding to  $G \cap F$  [12]. Consider a product structure on  $\mathcal{N}(F) \cong [-1, 1] \times F$ , such that  $G \cap \mathcal{N}(F) \cong [-1, 1] \times (G \cap F)$ . Take infinitely many copies of  $F$ , so that we have  $\{\pm 1/n, n \in \mathbb{N}\} \times F \subset \mathcal{N}(F)$ . Then we may “spin”  $G$  about  $F$  by adding  $G - (G \cap F)$  to infinitely many copies of  $F$ . We take  $G - (G \cap F) \oplus (\{\pm 1/n, n \in \mathbb{N}\} \times F)$  to get an infinite leaf  $L$  which spirals about  $\{0\} \times F \subset \mathcal{N}(F)$  (see Figure 2). Then  $(M \setminus \setminus F) \setminus \setminus L \cong I \times L$ , since it is obtained by adjoining to the product sutured manifold  $M \setminus \setminus (G \cup F)$  infinitely many copies of a product sutured manifold of the form  $I \times (F \setminus \setminus (G \cap F))$  along annuli. Thus,  $M \setminus \setminus F$  has a depth one taut oriented foliation.  $\square$

#### 4. NORMAL SURFACES

We’ll follow the notation of [35]. Let  $M$  be a compact 3-manifold with a triangulation  $\mathcal{T}$  with  $t$  tetrahedra. A normal surface  $F$  in  $M$  is a properly embedded surface in general position with the 1-skeleton  $\mathcal{T}^{(1)}$  and intersecting each tetrahedron  $\Delta$  of  $\mathcal{T}$  in properly embedded elementary disks which intersect each face in at most one straight line. A normal isotopy of  $M$  is an isotopy which leaves the simplices of  $\mathcal{T}$  invariant. If  $E$  is an elementary disk then  $\partial E$  is uniquely determined by the points  $E \cap \mathcal{T}^{(1)}$ , and a collection of normal disks in  $\Delta$  is uniquely determined up to normal isotopy by the number of points of  $E \cap \Delta^{(1)}$  in each edge of  $\Delta^{(1)}$ . A normal surface  $F$  is uniquely determined up to normal isotopy by the set of points  $F \cap \mathcal{T}^{(1)}$ . An elementary disk is determined, up to normal

FIGURE 2. Spinning  $G$  about  $F$  to obtain  $L$ 

isotopy, by the manner in which it separates the vertices of  $\Delta$  and we refer to the normal isotopy class of an elementary disk as its disk type. There are seven possible disk types in each tetrahedron corresponding to the seven possible separations of its four vertices; four of which consist of triangles and three consisting of quadrilaterals. The normal isotopy class of an arc in which an elementary disk meets a triangle of  $\Delta$  is called an arc type.

We fix an ordering  $(d_1, \dots, d_{7t})$  of the disk types in  $\mathcal{T}$  and assign a  $7t$ -tuple  $\mathbf{x} = (x_1, \dots, x_{7t})$ , called the normal coordinates of  $F$ , to a normal surface  $F$  by letting  $x_i$  denote the number of elementary disks in  $F$  of type  $d_i$ . The normal surface  $F$  is uniquely determined, up to normal isotopy, by its normal coordinates. Among  $7t$ -tuples of nonnegative integers  $\mathbf{x} = (x_1, \dots, x_{7t})$ , those corresponding to normal surfaces are characterized by two constraints. The first constraint is that it must be possible to realize the required 4-sided disk types  $d_i$  corresponding to nonzero  $x_i$ s by disjoint elementary disks. This is equivalent to allowing no more than one quadrilateral disk type to be represented in each tetrahedron. The second constraint concerns the matching of the edges of elementary disks along incident triangles of tetrahedra. Consider two tetrahedra meeting along a common 2-simplex and fix an arc type in this 2-simplex. There are exactly two disk types from each of the tetrahedra whose

elementary disks meet this 2-simplex in arcs of the given arc type. If the  $7t$ -tuple is to correspond to a normal surface then there must be the same number of elementary disks on both sides of the incident 2-simplex meeting it in arcs of the given type. This constraint can be given as a system of matching equations, one equation for each arc type in the 2-simplices of  $\mathcal{T}$  interior to  $M$ .

The matching equations are given by

$$x_i + x_j = x_k + x_l, \quad 0 \leq x_i,$$

where  $x_i, x_j, x_k, x_l$  share the same arc type in a 2-simplex of  $\mathcal{T}$ . The nonnegative solutions to the matching equations form an infinite linear cone  $\mathcal{S}_{\mathcal{T}} \subset \mathbb{R}^{7t}$ . The normalizing equation  $\sum_{i=1}^{7t} x_i = 1$  is added to form the system of normal equations for  $\mathcal{T}$ . The solution space  $\mathcal{P}_{\mathcal{T}} \subset \mathcal{S}_{\mathcal{T}}$  becomes a compact, convex, linear cell and is referred to as the projective solution space for  $\mathcal{T}$ . The projective class of a normal surface  $F$  is the image of the normal coordinates of  $F$  under the projection  $\mathcal{S}_{\mathcal{T}} \rightarrow \mathcal{P}_{\mathcal{T}}$ . A rational point  $\mathbf{z} \in \mathcal{P}_{\mathcal{T}}$  is said to be an admissible solution if corresponding to each tetrahedron there is at most one of the quadrilateral variables which is nonzero. Every admissible solution is the projective class of an embedded normal surface.

Given a normal surface  $F$ , we denote by  $C_F$  the unique minimal face of  $\mathcal{P}_{\mathcal{T}}$  that contains the projective class of  $F$ . A normal surface is carried by  $C_F$  if its projective class belongs to  $C_F$ . It is easy to see that a normal surface  $G$  is carried by  $C_F$  if and only if there exists a normal surface  $H$  such that  $mF = G + H$  for some positive integer  $m$ , where  $+$  indicates addition of normal surfaces, which is the unique way to perform cut and paste on  $G \cap H$  to obtain another normal surface when  $G$  and  $H$  have compatible quadrilateral types. Every rational point in  $C_F$  is an admissible solution, since it will have the same collection of quadrilateral types as  $F$ .

The weight of a normal surface  $F$  is the number of points in  $F \cap \mathcal{T}^{(1)}$ , which will be denoted by  $wt(F)$  (this was introduced in [10], [28], and [17] as a combinatorial substitute for the notion of the area of a surface). The type of a normal surface  $F$  is the collection of types of non-zero quadrilaterals and triangles in the surface  $F$ . Associated to a normal surface  $F$  is a branched surface  $\overline{B}_F$ , obtained by identifying normally parallel disks in each tetrahedron, and neighborhoods of arcs in each 2-simplex, and neighborhoods of vertices in each edge, and then perturbing to make generic (see Figure 3). For a branched surface  $B$  we will define  $Guts(B) = M \setminus B$ . If  $F$  is a normal surface with associated branched surface  $B$ , then  $Guts(B)$  consists of the non-normally product parts of  $M \setminus F$ , and gives a combinatorial substitute for  $Guts(M \setminus F)$ , which consists of the non-product pieces of the JSJ decomposition.

A taut surface  $F$  is said to be *lw-taut* (standing for *least weight-taut*) if it has minimal weight among all taut surfaces representing the homology class  $[F]$ . If  $F$  is lw-taut, then  $n$  pairwise disjoint copies of  $F$ , denoted by  $nF$ , is an lw-taut surface representing the class  $n[F]$ . Let  $\mathcal{B}_x(M) \subset H_2(M, \partial M; \mathbb{R})$  be the unit ball of the Thurston norm on homology. Tollefson and Wang prove that associated to each non-trivial homology class  $f \in H_2(M, \partial M; \mathbb{Z})$ , there is a face  $C_f$  of  $\mathcal{P}_{\mathcal{T}}$ , which is called a *complete lw-taut* face, which has the following properties ([35, Theorem 3.7]). Let  $[C_f]$  denote the set of homology



classes of all oriented normal surfaces carried by  $C_f$ . Then  $C_f$  carries every lw-taut normal surface representing any homology class in  $[C_f]$ . We extend the definition so that for  $\alpha \in H^1(M)$ ,  $C_\alpha = C_{PD(\alpha)}$ .

The construction of  $C_f$  is given in the following way. For a normal surface  $F$ , the unique minimal face  $C_F$  of  $\mathcal{P}_T$  carrying the projective class of  $F$  is said to be lw-taut if every surface carried by  $C_F$  is lw-taut. Tollefson and Wang prove that if  $F$  is lw-taut, then  $C_F$  is lw-taut [35, Theorem 3.3]. Then for  $f \in H_2(M, \partial M; \mathbb{Z})$ , one takes the finite set  $\{F_1, \dots, F_n\}$  up to normal isotopy of all oriented, lw-taut normal surfaces representing  $f$ , and let  $F = F_1 + \dots + F_n$  be the lw-taut surface representing the homology class  $nf$ , which is canonically associated to  $f$  up to normal isotopy. The addition here is both addition of normal surfaces and oriented cut-and-paste addition, which preserves the homology class. Then define  $C_f \equiv C_F$ , which will have the desired properties. One may also associate the branched surface  $\overline{B}_f \equiv \overline{B}_F$  canonically to the homology class  $f$ .

If  $F$  is a lw-taut surface, then there is canonically associated a normal branched surface  $B_F$ , which is obtained from  $\overline{B}_F$  by splitting along punctured disks of contact. Tollefson and Wang prove that  $B_F$  is a Reebless incompressible branched surface which is oriented. If  $F$  is the canonical lw-taut normal surface associated to  $f$ , then  $B_f \equiv B_F$  has the property that every surface carried by  $C_F$  is isotopic to a surface carried by  $B_f$ , such that the orientation on the surface is induced by the orientation of  $B_f$ . This follows from the proof of [35, Lemma 6.1], where they prove that every surface carried by  $C_F$  is homologous to a surface carried by  $B_f$ , by cut-and-paste of normal disks. However, in the case that  $M$  is irreducible, cut-and-paste of disks may be achieved by an isotopy. The branched surface  $\overline{B}_F$  associated to  $F$  has a fibered neighborhood  $\overline{N}_F = \overline{N}_f$ , and  $B_F$  has a fibered neighborhood  $N_F = N_f$ . Moreover, we may isotope  $B_F$  to be normally carried by  $\overline{N}_F$  so that it is transverse to the fibers. In this case  $Guts(B_f) = M \setminus N_f$  is a taut-sutured manifold, with  $\gamma(B_f) = \partial_v(N_f)$ , and  $R(\gamma) = \partial_h(N_f)$ , such that the orientation of each component of  $R(\gamma)$  is induced by the orientation of  $B_f$ . From the construction, suppose that  $g \in [C_f]$ , then  $C_g \subseteq C_f$ ,  $[C_g] \subseteq [C_f]$ , and  $\overline{B}_g \subseteq \overline{B}_f$  (up to normal isotopy). For each component  $Q$  of  $Guts(\overline{B}_f)$ , there is a complexity  $c(Q)$  which is the number of normal disk types  $D$  such that  $D$  is compatible with the disk types of  $C_f$ , and  $D$  cannot be normally isotoped into  $\overline{N}_f$  transverse to the fibers to miss  $Q$  (see Figure 3). Define

$$c(Guts(\overline{B}_f)) = \max\{c(Q) | Q \text{ is a component of } Guts(\overline{B}_f)\}.$$

**Lemma 4.1.** *Let  $M$  be a compact irreducible orientable 3-manifold. Suppose that  $f \in H_2(M, \partial M)$  is a homology class lying in the interior of a cone  $E$  over a face of  $\mathcal{B}_x(M)$ . Moreover assume that  $F \in \text{int}(C_f)$ , where  $F$  is the canonical projective representative of the homology class  $f$  introduced above, and  $C_f$  is a maximal lw-taut face. Let  $B_f$  be the taut oriented normal branched surface associated to the cone  $C_f$ , and let  $Guts(B_f)$  be the complementary regions of  $N_f$ . Then  $\text{im}\{H^1(M) \rightarrow H^1(Guts(B_f))\} = 0$ .*

*Proof.* Suppose that  $\text{im}\{H^1(M) \rightarrow H^1(Guts(B_f))\} \neq 0$ . Let  $g \in H_2(M, \partial M; \mathbb{Z})$  be a homology class dual to a cohomology class  $PD(g) \in H^1(M)$  with non-zero image in  $H^1(Guts(B_f))$ . Then for large enough  $k$ , we have  $kf + g \in E$ , since  $f$  lies in the interior of  $E$ , and we may assume that  $kf + g \in [C_f]$ . But then any lw-taut surface  $S$  such that  $[S] = kf + g$  is isotopic to one carried by  $B_f$  by Lemma 6.1 [35]. This gives

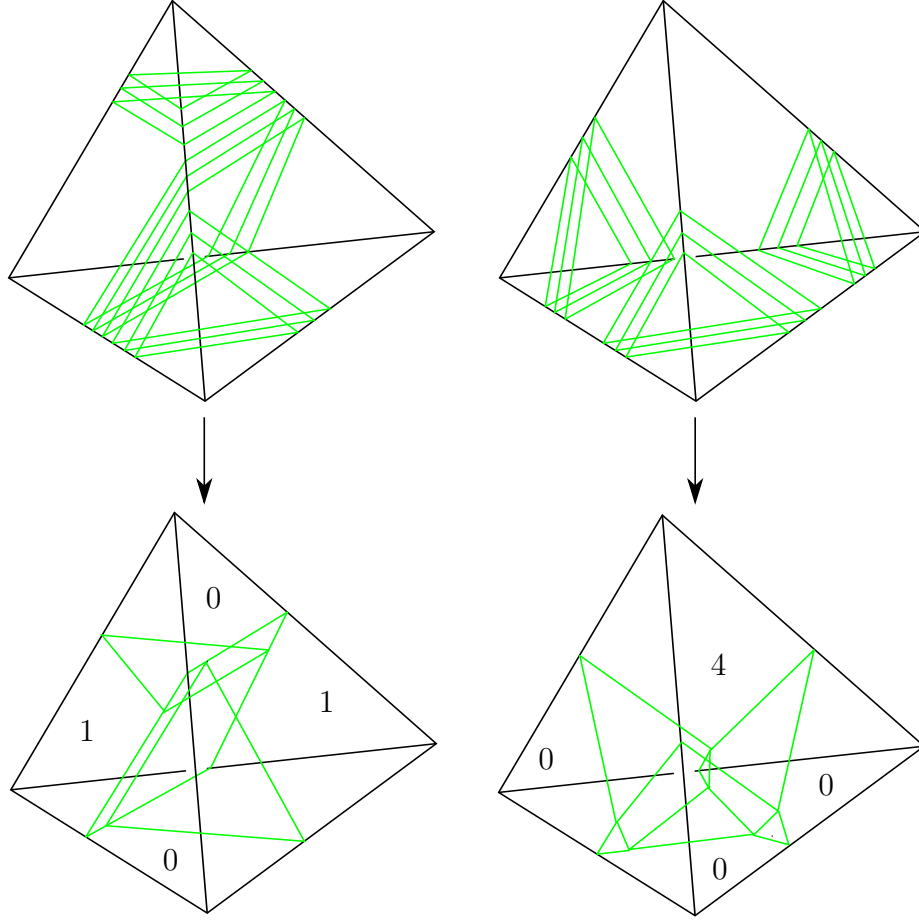


FIGURE 3. Local contribution to complexity of the guts of a branched surface obtained by pinching normal disks

a contradiction, since for any surface  $S$  carried by  $B_f$ , we must have for any oriented loop  $\alpha \subset \text{Guts}(B_f)$ ,  $\alpha \cap S = \emptyset$ . But there is an oriented loop  $\alpha \subset \text{Guts}(B_f)$  such that  $PD(g)(\alpha) \neq 0$ , or equivalently  $g \cap [\alpha] \neq 0$ . But since  $f \cap [\alpha] = 0$ , this implies that  $[S] \cap [\alpha] = (nf + g) \cap [\alpha] = g \cap [\alpha] \neq 0$ , a contradiction.  $\square$

**Corollary 4.2.** *Under the same hypotheses as Lemma 4.1,  $\text{im}\{H^1(M) \rightarrow H^1(\text{Guts}(\overline{B}_f))\} = 0$ .*

*Proof.* The branched surface  $B_f$  is obtained from  $\overline{B}_f$  by splitting along punctured disks of contact. Thus, we have an embedding  $\text{Guts}(\overline{B}_f) \subset \text{Guts}(B_f)$ , which gives a factorization  $H^1(M) \rightarrow H^1(\text{Guts}(B_f)) \rightarrow H^1(\text{Guts}(\overline{B}_f))$ . Then we apply Lemma 4.1 to see that the image is 0.  $\square$

**Lemma 4.3.** *Suppose that  $f \in H_2(M, \partial M; \mathbb{Z})$  is a homology class such that  $(\text{Guts}(B_f), \gamma(B_f))$  is a product sutured manifold. Then  $f$  is a fibered homology class.*

*Proof.* Let  $F$  be an lw-taut surface fully carried by  $B_f$  representing  $nf$  as above. Associated to  $B_f$  is an  $I$ -bundle  $\overline{L}_F$  as described in Section 6 of [35], which is obtained by splitting

$\overline{N}_F$  along  $2F - \text{int}_{2F}(\partial_h \overline{N}_F)$ . If  $(\text{Guts}(B_f), \gamma(B_f))$  is a product sutured manifold, then the  $I$ -bundle structures on  $\overline{L}_F \cup \text{Guts}(B_f)$  match together to give an  $I$ -bundle structure on the complement of  $2F$ , which implies that  $F$  is a fiber.  $\square$

## 5. VIRTUAL FIBERING

The following criterion for virtual fibering is the principal result in this paper.

**Theorem 5.1.** *Let  $M$  be a connected orientable irreducible 3-manifold with  $\chi(M) = 0$  such that  $\pi_1(M)$  is RFRS. If  $0 \neq f \in H^1(M)$  is a non-fibered homology class, then there exists a finite-sheeted cover  $p_{i,0} : M_i \rightarrow M$  such that  $p_{i,0}^* f \in H^1(M_i)$  lies in the cone over the boundary of a fibered face of  $\mathcal{B}_x(M_i)$ .*

*Proof.* Let  $G = \pi_1(M)$ , with triangulation  $\mathcal{T}$ . We are assuming that  $G$  is RFRS, so there exists  $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots$  such that  $G \triangleright G_i$ ,  $[G : G_i] < \infty$ , and  $G_{i+1} > (G_i)_r^{(1)}$  so that  $G_i/G_{i+1}$  is abelian. Let  $M_i$  be the regular cover of  $M$  corresponding to the subgroup  $G_i$ , so that  $M_0 = M$ . First, we give an outline of the argument. Choose a norm-minimizing surface  $\Sigma_0$  in  $M_0$  so that  $[\Sigma_0]$  lies in a maximal face of  $\mathcal{B}_x(M_0)$ , and whose projective class is a small perturbation of  $PD(f)$ . Each component of the guts of  $\Sigma_0$  will lift to  $M_1$  by Lemma 4.1. We successively lift each component of the guts of  $\Sigma_0$  to  $M_i$  until a component of the guts has a non-trivial image in  $H^1(M_i)$ . We perturb the projective homology class of the preimage of  $\Sigma_0$  in  $M_i$  again, to get a surface with simpler gut regions. The new gut regions will be obtained by sutured manifold decomposition from the original gut regions of  $\Sigma_0$ , and in fact they embed back down into  $M$ . The technical heart of the proof is the machinery to show that these sutured manifold decompositions terminate. Gabai showed that any particular taut sutured manifold hierarchy will terminate, but his complexity does not bound a priori the number of possible decompositions. Since we are in some sense performing multiple sutured manifold decompositions “in parallel” in each cover, it is not guaranteed that the family of decompositions will terminate. Thus we use the machinery of normal surfaces. The complexity measures roughly how many normal disks types are still available in each gut region to decompose along.

For  $j > i$ , let  $p_{j,i} : M_j \rightarrow M_i$  be the covering map, and let  $\mathcal{T}_i = p_{i,0}^{-1}(\mathcal{T})$  be the induced triangulation of  $M_i$ . Let our given cohomology class  $f \in H^1(M_0)$  be denoted by  $f_0$ . We construct by induction cohomology classes  $f_i \in H^1(M_i)$ ,  $i > 0$  such that the canonical lw-taut surface  $F_i$  such that  $[F_i] = n_i PD(f_i)$  lies in the interior of  $C_{f_i}$ . Given  $f_i$ , choose  $f_{i+1} \in H^1(M_{i+1})$  to satisfy the following properties:

- the projective class of  $f_{i+1}$  is a small perturbation of  $p_{i+1,i}^* f_i$  in the unit norm ball  $\mathcal{B}_x(M_{i+1})$
- $C_{p_{i+1,i}^* f_i} \subset C_{f_{i+1}}$  is a facet
- $f_{i+1}$  lies in a maximal face of the unit norm ball  $\mathcal{B}_x(M_{i+1})$
- there is a lw-taut surface representative  $F_{i+1}$  of  $n_{i+1} PD(f_{i+1})$  so that  $F_{i+1} \in \text{int}(C_{f_{i+1}})$  and  $C_{f_{i+1}}$  is a maximal face.

This concludes the inductive construction of the cohomology classes  $f_i$ . We now prove that for  $i$  large enough,  $f_i$  is a fibered cohomology class in  $H^1(M_i)$ .

To prove this, we show that if  $f_i$  is not a fibered cohomology class, then there exists  $J > i$  such that  $c(\text{Guts}(\overline{B}_{f_J})) < c(\text{Guts}(\overline{B}_{f_i}))$ . By Lemma 4.1 we have  $\text{im}\{H^1(M_i) \rightarrow$

$H^1(Guts(B_{f_i}))\} = 0$ . Thus each component of  $Guts(B_{f_i})$  lifts to  $M_{i+1}$ . Let  $Q \subseteq Guts(B_{f_i})$  be a connected component such that  $\pi_1(Q) \neq 1$ . Recall that since  $B_{f_i}$  is a Reebless incompressible branched surface,  $\pi_1 Q \hookrightarrow \pi_1 M_i$  is injective [27]. Let  $\overline{Q} = Q \cap Guts(\overline{B}_{f_i})$ , where we are assuming that  $B_{f_i} \subset \overline{N}_{f_i}$  is carried transverse to the fibers. Since  $\cap_j G_j = 1$ , there exists  $j = j(Q) > i$  such that  $G_{j+1} \cap \pi_1(Q) \neq \pi_1(Q) \leq G_j$  (since  $G_{j+1} \triangleleft G$ , notice that we do not need to worry about the conjugacy class of  $\pi_1 Q < \pi_1 M = G$ ). Then there exists  $g \in \pi_1(\overline{Q})$  such that the projection of  $g$  to the abelian group  $G_j/G_{j+1}$  is non-trivial. Therefore  $g$  represents a homologically non-trivial element in  $H_1(M_j)$  of infinite order. If  $F_i \subset M_i$  is an lw-taut surface such that  $[F_i] = n_i PD(f_i)$ , then the preimage  $p_{j,i}^{-1}(F_i) \subset M_j$  of  $F_i$  representing the homology class  $n_i PD(p_{j,i}^* f_i) \in H_2(M_j, \partial M_j)$  is also an lw-taut surface by [34, Theorem 3.2]. We see that the preimage  $p_{j,i}^{-1}(\overline{B}_{f_i}) = \overline{B}_{p_{j,i}^* f_i} \subset \overline{B}_{f_j} \subset M_j$ .

Thus,  $Q$  will lift to  $M_j$ , but  $\overline{Q}$  will be decomposed by  $\overline{B}_{f_j}$  into  $\overline{Q}_j$  for which  $c(\overline{Q}_j) < c(\overline{Q})$ . Otherwise  $\overline{Q}$  is a component of  $Guts(\overline{B}_{f_j})$ , which would contradict Corollary 4.2, since  $im\{H^1(M_j) \rightarrow H^1(Q)\} \neq 0$ . Since  $\overline{B}_{f_j}$  is obtained by identifying parallel normal disks, and  $\overline{B}_{p_{j,i}^* f_i} \subset \overline{B}_{f_j}$ , there must be a normal disk type of  $C_{f_j}$  which does not appear as a normal disk type of  $C_{p_{j,i}^* f_i}$ , and thus “kills” the lift of  $g$  from  $M_i$  to  $M_j$ . In particular, this normal disk must lie inside of the lift of  $\overline{Q}$  to  $M_j$ , in such a way that it may not be normally isotoped to lie inside of  $\overline{N}_{p_{j,i}^* f_i}$ . Since  $M_j$  is a regular cover of  $M_i$ , this implies that the same holds for every lift of  $\overline{Q}$  to  $M_j$ . Now, we choose  $J$  to be the maximum of  $j(Q)$  over all components  $Q$  of  $Guts(B_i)$ . Thus,  $c(Guts(\overline{B}_J)) < c(Guts(\overline{B}_i))$ .

If  $\pi_1(Q) = 1$ , then  $Q$  must be a taut sutured ball. If every component  $Q$  of  $Guts(B_{f_i})$  is a taut-sutured ball, then by Lemma 4.3,  $f_i$  is a fibered cohomology class. By the induction above, we see that there exists  $I$  such that for all  $i \geq I$ ,  $c(Guts(\overline{B}_i)) = c(Guts(\overline{B}_I))$ . But this implies that for every component  $Q$  of  $Guts(B_I)$ ,  $\pi_1(Q) = 1$ , and therefore  $f_I$  is a fibered cohomology class.  $\square$

The following theorem is a corollary of Theorem 5.1 and Corollary 2.3.

**Theorem 5.2.** *If  $\mathcal{O}$  is a 3-dimensional hyperbolic reflection orbifold of finite volume or an arithmetic hyperbolic orbifold defined by a quadratic form, then there exists an orientable manifold cover  $\tilde{\mathcal{O}} \rightarrow \mathcal{O}$  such that  $\tilde{\mathcal{O}}$  fibers over  $S^1$ .*

**Remark:** Arithmetic groups defined by quadratic forms include the Bianchi groups  $PSL(2, \mathcal{O}_d)$ , where  $\mathcal{O}_d$  is the ring of integers in a quadratic imaginary number field  $\mathbb{Q}(\sqrt{-d})$  where  $d$  is a positive integer, and the Seifert-Weber dodecahedral space. These examples are some of the oldest known examples of hyperbolic lattices.

## 6. VIRTUAL DEPTH ONE

The next theorem gives an analogue of Theorem 5.1 for taut sutured manifolds. A taut sutured manifold  $(M, \gamma)$  with  $\chi(M) < 0$  virtually fibers over an interval if and only if it is a product sutured manifold. If  $(M, \gamma)$  has a depth one taut-oriented foliation  $\mathcal{F}$ , then  $\mathcal{F}_{\text{int}M}$  gives a fibering of  $\text{int}M$  over  $S^1$  if the corresponding cohomology class is rational. Thus, depth one seems to be the best suited analogue to fibering for taut sutured manifolds with boundary.

**Theorem 6.1.** *Let  $(M, \gamma)$  be a connected taut sutured manifold such that  $\pi_1(M)$  satisfies the RFRS property. Then there is a finite-sheeted cover  $(\tilde{M}, \tilde{\gamma}) \rightarrow (M, \gamma)$  such that  $(\tilde{M}, \tilde{\gamma})$  has a depth one taut oriented foliation.*

*Proof.* The proof is similar to the proof of Theorem 5.1. We double  $M$  along  $R(\gamma)$  to get a manifold  $DM_\gamma$  with (possibly empty) torus boundary, which admits an orientation reversing involution  $\tau$  exchanging the two copies of  $M$ , and fixing  $R_\pm(\gamma)$  while reversing their coorientations. In order that we may apply the lw-taut technology, we choose a triangulation  $\mathcal{T}$  of  $DM_\gamma$  for which the only lw-taut surfaces representing  $[R_\pm(\gamma)]$  are isotopic to  $R_\pm(\gamma)$ . This is possible by choosing a triangulation of  $DM_\gamma$  for which  $R_\pm(\gamma)$  are normal of the same weight and such that  $R_+(\gamma) \cup R_-(\gamma)$  intersects each tetrahedron in at most one normal disk, and each tetrahedron meeting  $R_\pm(\gamma)$  does not intersect any other tetrahedron meeting  $R_\pm(\gamma)$  except in the faces containing arcs of a normal disk of  $R_\pm(\gamma)$ . In this case, if another normal surface is homologous to  $R_\pm(\gamma)$ , then it must meet a tetrahedron which does not meet  $R_+(\gamma) \cup R_-(\gamma)$ , since the tetrahedra meeting  $R_\pm(\gamma)$  form a product neighborhood. Now, stellar subdivide the tetrahedra not meeting  $R_\pm(\gamma)$  a number of times, until every normal surface  $F$  homologous to  $R_\pm(\gamma)$  has weight  $wt(F) > wt(R_\pm(\gamma))$ . This is the triangulation that we work with on  $DM_\gamma$  in order that we may apply the lw-taut machinery without modification.

The retraction  $r : DM_\gamma \rightarrow M$  induces a retraction  $r_\# : \pi_1 DM_\gamma \rightarrow \pi_1 M$ . Choose orientations on the surfaces  $R_\pm(\gamma)$  so that they are homologous, and call the resulting lw-taut surface  $F$ . Then  $[F] = 2[R_+]$ . Also,  $B_F = \overline{B}_F = F$ , and  $Guts(B_F) = DM_\gamma \setminus \setminus F$ . Notice that for each component  $Q$  of  $Guts(B_F)$ ,  $\pi_1(Q) < \pi_1(M)$ , up to the action of  $\tau$ . We proceed now as in the proof of Theorem 5.1. There are covers  $M_i \rightarrow M$  satisfying the RFRS criterion. We construct covers  $N_i \rightarrow DM_\gamma$  so that  $\pi_1(N_i) = r_\#^{-1}(\pi_1(M_i))$ , that is induced by the retraction  $r : DM_\gamma \rightarrow M$ . We may apply the proof of Theorem 5.1 to find a cover  $p : N_i \rightarrow DM_\gamma$  and a cohomology class  $f \in H^1(N_i)$  which is a perturbation of  $p^*(PD([F]))$  such that  $f$  is a fibered class, by induction on  $c(Guts(\overline{B}_{f_i}))$ . We also have that each component of  $N \setminus p^{-1}(F)$  is a cover of  $M$ . By Theorem 3.7,  $N \setminus p^{-1}(F)$  has a depth one taut oriented foliation. Thus,  $M$  has a finite sheeted cover with a depth one taut oriented foliation.  $\square$

The following theorem is a corollary of Corollary 2.3 and Theorem 6.1.

**Theorem 6.2.** *Let  $(M, \gamma)$  be a taut sutured compression body. Then there is a finite-sheeted cover  $(\tilde{M}, \tilde{\gamma}) \rightarrow (M, \gamma)$  such that  $(\tilde{M}, \tilde{\gamma})$  has a depth one taut oriented foliation.*

**Remark:** The previous theorem is non-trivial, since there exist examples due to Brittenham of taut sutured genus 2 handlebodies which admit no depth one taut oriented foliation [6].

## 7. FIBERED FACES

In this section, we prove a result of Long and Reid [23], which says that a fibered arithmetic 3-manifold has finite index covers with arbitrarily many fibered faces of the Thurston norm unit ball. The first result of this type was proved by Dunfield and Ramakrishnan for a specific arithmetic 3-manifold [9]. It follows from Theorem 5.2 that this holds for

any arithmetic 3-manifold defined by a quadratic form (without the added assumption of fibering), although this will also follow from Theorem 7.2. The proof is inspired by the strategy of Long and Reid's argument, where we replace their use of pseudo-Anosov flows with harmonic 2-forms.

**Theorem 7.1.** *For any  $k > 0$ , an arithmetic 3-manifold  $M$  which fibers over  $S^1$  has a finite-sheeted cover  $M_k \rightarrow M$  such that  $\mathcal{B}_x(M_k)$  has at least  $k$  distinct fibered faces.*

*Proof.* For simplicity, assume that  $M$  is closed. Let  $\alpha \in H_2(M)$  be a homology class which fibers. Then there is an Euler class  $e \in H^2(M)$  such that for any  $\beta \in H_2(M)$  lying in the same face of  $\mathcal{B}_x(M)$  as  $\alpha$ , one has  $e(\beta) = x(\beta)$  [33]. Consider a harmonic representative  $e_0 \in \mathcal{H}^2(M)$ , where  $\mathcal{H}^2(M) \cong H^2(M; \mathbb{R})$  is the space of harmonic 2-forms on  $M$  with respect to the hyperbolic metric. Let  $\tilde{e}_0 \in \mathcal{H}^2(\mathbb{H}^3)$  be the image of  $e_0$  in the universal cover  $\mathbb{H}^3 \rightarrow \mathbb{H}^3/\Gamma = M$ . By the proof of [2, Theorem 0.1], the orbit of  $\tilde{e}_0$  under  $Comm(\Gamma)$  is infinite. We finish again as in the proof of [2, Theorem 0.1]. For any  $k$ , choose  $g_1, \dots, g_k \in Comm(\Gamma)$  such that  $\{g_{j*}(\tilde{e}_0) | j = 1, \dots, k\}$  are distinct. Then let  $\Gamma_k = \cap_j (g_j \Gamma g_j^{-1})$ . The 2-forms  $g_{j*} \tilde{e}_0$  project to distinct harmonic 2-forms on  $M_k = \mathbb{H}^3/\Gamma_k$ , each of which is dual to a distinct fibered face of the Thurston norm unit ball which is the preimage of a fibered face on  $\mathbb{H}^3/g_j \Gamma g_j^{-1} \cong M$  associated to the projection of  $g_{j*} \tilde{e}_0$ .  $\square$

Next, we prove that an irreducible 3-manifold  $M$  with  $\chi(M) = 0$  and  $\pi_1(M)$  RFRS has virtually infinitely many fibered faces.

**Theorem 7.2.** *Let  $M$  be a compact orientable irreducible 3-manifold with  $\chi(M) = 0$  and  $\pi_1(M)$  virtually RFRS and not virtually abelian. Then for any  $k > 0$ , there exists a cover  $M_k \rightarrow M$  such that  $\mathcal{B}_x(M_k)$  has at least  $k$  distinct fibered faces.*

*Proof.* Since  $\pi_1(M)$  is RFRS, we may assume that  $M$  fibers by Theorem 5.1. We will prove the result by induction. For  $k > 0$ , assume  $M$  has a cover  $M_k \rightarrow M$  such that  $\mathcal{B}_x(M_k)$  has  $\geq k$  fibered faces. Since  $\pi_1(M)$  is RFRS and not virtually abelian, there exists a cover  $M'_k \rightarrow M_k$  such that  $\beta_1(M'_k) > \beta_1(M_k)$  and  $\mathcal{B}_x(M'_k) \subset H^1(M'_k)$  has more faces than  $\mathcal{B}_x(M_k) \subset H^1(M_k)$ . If  $\mathcal{B}_x(M'_k)$  has all fibered faces, then  $M'_k$  has  $> k$  fibered faces since by hypothesis it has more faces than  $\mathcal{B}_x(M_k)$ . Otherwise,  $\mathcal{B}_x(M'_k)$  has a non-fibered face  $U$ , such that there is an Euler class  $e \in H^2(M'_k)$  so that  $x(z) = e(PD(z))$  for  $z \in U$ . Fix  $z \in \text{int}(U)$  and take  $p : M''_k \rightarrow M'_k$  such that  $p^*z$  lies in the boundary of a fibered face  $U'$  of  $\mathcal{B}_x(M''_k)$  by Theorem 5.1. Let  $e' \in H^2(M''_k)$  be the Euler class associated to  $U'$ . Then  $e'(PD(p^*z)) = p^*(e)(PD(p^*z))$ . But then for any other fibered face  $V$  of  $\mathcal{B}_x(M'_k)$  with associated Euler class  $f$ , we have  $x(z) \neq f(PD(z))$ . Thus,  $x(PD(p^*z)) \neq p^*(f)(PD(p^*z))$ , and therefore  $e' \neq p^*f$ . So  $U'$  must be a fibered face not containing  $p^*(V)$ , and therefore  $\mathcal{B}_x(M''_k)$  has at least  $k + 1$  fibered faces.

Geometrically, what is happening here is that the polyhedron  $p^*\mathcal{B}_x(M'_k)$  is the intersection of the subspace  $p^*H^1(M'_k) \subset H^1(M''_k)$  with  $\mathcal{B}_x(M''_k)$ . Each fibered face of  $p^*\mathcal{B}_x(M'_k)$  will lie in the interior of a fibered face of  $\mathcal{B}_x(M''_k)$ , whereas the non-fibered face  $p^*U$  will lie in a lower dimensional skeleton of  $\mathcal{B}_x(M''_k)$  in the boundary of a fibered face. The Euler classes used above are linear functionals dual to each face, and therefore are useful for distinguishing the faces of the unit norm ball.  $\square$

## 8. APPLICATIONS

The following result was observed in [20, Section 5] to follow if arithmetic 3-manifolds defined by quadratic forms are virtually fibered.

**Theorem 8.1.** *All arithmetic uniform lattices of simplest type (i.e. defined by a quadratic form) with  $n \geq 4$  are not coherent.*

A group is *FD* if the finite representations are dense in all unitary representations with respect to the Fell topology. The following result follows from [25, Corollary 2.5, Lemma 2.6]. This generalizes [25, Theorem 2.8 (1)], which proves that  $\mathrm{SL}(2, \mathbb{Z}[i])$  and  $\mathrm{SL}(2, \mathbb{Z}[\sqrt{-3}])$  have property FD. The following theorem was pointed out by Alan Reid to be a corollary of Theorem 5.2.

**Theorem 8.2.** *Arithmetic Kleinian groups defined by quadratic forms and reflection groups are FD.*

## 9. CONCLUSION

It seems natural to extend Thurston's conjecture to the class of aspherical 3-manifolds which have at least one hyperbolic piece in the geometric decomposition.

**Conjecture 9.1.** *If  $M$  is an irreducible orientable compact 3-manifold with  $\chi(M) = 0$  which is not a graph manifold, then there is a finite-sheeted cover  $\tilde{M} \rightarrow M$  such that  $\tilde{M}$  fibers over  $S^1$ .*

**Remark:** There are closed graph manifolds which do not virtually fiber over  $S^1$ . The classic examples are Seifert fibered spaces for which the rational Euler class of the Seifert fibration is non-zero and the base orbifold has Euler characteristic non-zero [29]; see also [26, Theorem F] for examples of irreducible graph manifolds which do not virtually fiber. If an irreducible manifold with a non-trivial geometric decomposition fibers over  $S^1$ , then each of the geometric factors of the torus decomposition will also fiber, so it is certainly necessary for the above conjecture to be true that all of the geometric pieces in the torus decomposition virtually fiber. However, the obstructions to virtual fibering vanish when a graph manifold has boundary or a metric of non-positive curvature [8]. It would be interesting to know if a graph manifold  $M$  which virtually fibers has  $\pi_1 M$  virtually RFRS.

As mentioned before, the following conjecture is a natural analogue of the virtual fibering conjecture for taut sutured manifolds.

**Conjecture 9.2.** *Let  $(M, \gamma)$  be a taut sutured manifold such that  $\chi(M) < 0$ . Then there is a finite-sheeted cover  $(\tilde{M}, \tilde{\gamma}) \rightarrow (M, \gamma)$  such that  $(\tilde{M}, \tilde{\gamma})$  has a depth one taut oriented foliation.*

The following conjecture is a natural generalization of the virtual fibering conjecture. Once a manifold  $M$  fibers and  $\beta_1(M) > 1$ , then  $M$  will fiber over the circle in infinitely many different ways. But the fibrations coming from a single face of the Thurston norm are closely related to each other, so one can ask how many unrelated fiberings are there? Clearly, the following conjecture is stronger than Conjecture 9.1.

**Conjecture 9.3.** *If  $M$  is irreducible, compact orientable with  $\chi(M) = 0$  and not a graph manifold, then for any  $k > 0$ , there is a finite index cover  $M_k \rightarrow M$  with  $k$  fibered faces of  $\mathcal{B}_x(M_k)$ .*

**Question:** Which 3-manifolds have RFRS or virtually RFRS fundamental group? Clearly Seifert fibered spaces with non-zero Euler class of fibration and non-zero Euler characteristic of base surface cannot have any finite cover with RFRS fundamental group. Boileau and Wang have examples derived from this which cannot have RFRS fundamental group [4]. Sol and Nil manifolds are virtually fibered, but not virtually RFRS. Also, knot complements cannot have RFRS fundamental group, but it is still possible that they are virtually RFRS. Every hyperbolic 3-manifold has a finite sheeted cover which is pro- $p$  [24], but in general a tower of  $p$ -covers will not satisfy the rationality condition of the RFRS definition. One strategy to try to prove Conjectures 9.1 and 9.3 would be to show that an irreducible 3-manifold  $M$  which is not a graph manifold has  $\pi_1(M)$  virtually a subgroup of a right-angled Coxeter group, and therefore is virtually RFRS. As noted in [15], this is equivalent to showing that  $\pi_1(M)$  is LERF and that  $\pi_1(M)$  acts properly and cocompactly on a CAT(0) cube complex. Having such an action would follow from the existence of enough immersed  $\pi_1$ -injective surfaces to bind every element of  $\pi_1 M$ .

It would be interesting to try to compute the index of the cover of a Bianchi group for which the group will virtually fiber. Similarly for all-ideal right-angled reflection groups, for which we suspect there is a very small finite-index cover which fibers, possibly a four-fold manifold cover.

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